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Wishart and anti-Wishart random matrices

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Abstract

We provide a compact exact representation for the distribution of the matrix elements of the Wishart-type random matrices $\mathcal{A}^\dagger\mathcal{A}$, for any finite number of rows and columns of $\mathcal{A}$, without any large N approximations. In particular, we treat the case when the Wishart-type random matrix contains redundant, non-random information, which is a new result. This representation is of interest for a procedure for reconstructing the redundant information hidden in Wishart matrices, with potential applications to numerous models based on biological, social and artificial intelligence networks.

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1. Introduction

Random matrices of the form $\Omega = \mathcal{A}^\dagger\mathcal{A}$, where $\mathcal{A}$ is a random rectangular matrix of size $n \times k$, occur in many applications. Introduced in the classical paper by Wishart [1] (probably the first application of random matrix models), they form the cornerstone of multivariate statistical analysis [2]. Standard applications include biology, economy, telecommunication, to mention a few. Usually, n samples of k-dimensional rows of data are used to construct the covariance matrix. Typical measurements are, e.g., samples of large numbers of meteorological observations at various sites collected at time intervals [3], high-frequency financial data for large portfolios [4] or wireless channels with multiple antennae or receivers [5]. Wishart matrices appear also in fundamental science, ranging from condensed matter physics [6], nuclear physics [7] to chiral quantum chromodynamics [8]. Recently, the topic of retrieving the redundant information from Wishart matrices got a new twist due to the spectacular increase in computing and storage powers. The key problem is to handle effectively large incidence matrices constructed by intelligent retrieval (IR) engines. These new challenges vary from eliminating the redundant information in internet traffic, through exploiting the hidden information of knowledge networks [9] to unravelling the cross-correlations in bioinformatics and genetics.

Let us consider a matrix $\Omega = \mathcal{A}^\dagger\mathcal{A}$. There are two qualitatively different situations that may appear. When $n \geqslant k$, all the elements of Ω are indeed random. Their probability distribution was first derived by Wishart in [1]. In the opposite case $n < k$, there is a lot of redundancy in Ω, and only a part of the matrix elements are random, the remaining elements are unambiguously determined in terms of the random ones. The analytical expression for the probability distribution of matrix elements of Ω in this 'anti-Wishart' case (this name was coined by Yi-Cheng Zhang) remained, however, unknown. The numerical study of the anti-Wishart case was presented in [9]. In this paper, we would like to derive this probability distribution as well as to provide a procedure for reconstructing the redundant information from the first n rows of the matrix. All these results are exact for any n and k, without any large N approximations.

The plan of this paper is as follows. In the next two sections we will derive recurrence relations for the probability distributions for *complex* Gaussian matrices $\mathcal{A}$. In section 4, we will use these relations to rederive the classical Wishart case, and then proceed to analyse the new anti-Wishart case. For completeness, in section 5 we will state some known results for the joint eigenvalue distributions in the anti-Wishart case. We will conclude the paper with a discussion and two appendices which state the analogous results, when $\mathcal{A}$ are *real* Gaussian matrices and include some mathematical details.

2. Wishart and anti-Wishart random matrices

Let $\mathcal{A}$ be a *complex* rectangular matrix of size $n \times k$ taken from a Gaussian ensemble:

$$P(\mathcal{A}) = \frac{1}{\pi^{nk}} \mathrm{e}^{-\mathrm{Tr}\,\mathcal{A}^\dagger\mathcal{A}}. \tag{1}$$

We would like to derive the probability distribution of the elements of the Wishart matrix $\Omega = \mathcal{A}^\dagger\mathcal{A}$. We thus have to evaluate

$$P_{n,k}(\Omega) = \int \mathrm{d}\mathcal{A}\,\mathrm{d}\mathcal{A}^\dagger\,\delta(\Omega - \mathcal{A}^\dagger\mathcal{A})P(\mathcal{A}). \tag{2}$$

When $n \geqslant k$ ('Wishart case') the resulting distribution was obtained by Wishart after a quite intricate calculation for real $\mathcal{A}$ (see, e.g., [2]). The analogous result for complex matrices is

$$P_{n,k}^{\mathrm{Wishart}}(\Omega) = C_{n,k}(\det\Omega)^{n-k}\,\mathrm{e}^{-\mathrm{Tr}\,\Omega} \tag{3}$$

where $C_{n,k}$ is a normalization constant.

When $n < k$ ('anti-Wishart case') no such explicit formula was known. The goal of this work is to give a simple unified derivation of $P_{n,k}(\Omega)$ which works in both cases and to provide a procedure for reconstructing the redundant information from the first n rows. The problem of reconstructing the Ω matrix from more realistic sparse data will be considered in a subsequent work.

As was noted in [9], determination of $P_{n,k}(\Omega)$ can be easily translated, using the integral representation

$$\delta(\Omega - \mathcal{A}^\dagger\mathcal{A}) = \frac{1}{(2\pi)^{k^2}} \int \mathrm{d}\mathcal{T}\,\mathrm{e}^{\mathrm{i}\mathrm{Tr}\,\mathcal{T}(\Omega - \mathcal{A}^\dagger\mathcal{A})} \tag{4}$$

into the integral

$$P_{n,k}(\Omega) = C'_{n,k} \int \mathrm{d}\mathcal{T}\,\det(1 + \mathrm{i}\mathcal{T})^{-n}\,\mathrm{e}^{\mathrm{i}\mathrm{Tr}\,\mathcal{T}\Omega} \tag{5}$$

where $\mathcal{T}$ is a $k \times k$ *Hermitian* matrix.

3. Recurrence relations

The integrals of the form (5) with Ω diagonal were considered for $n \geqslant k$ in an interesting paper by Fyodorov [10]. Here we will slightly generalize his procedure for arbitrary non-diagonal Ω. Although the integrals (5) are invariant with respect to unitary transformations, this generalization is necessary in the case $n < k$. Indeed, then the matrix Ω has $k - n$ *exact* zero eigenvalues and the Jacobian for the diagonalization $\Omega = U\Lambda U^\dagger$ will be quite nontrivial.

Let us first decompose the matrices $\mathcal{T} \equiv \mathcal{T}_k$ and $\Omega \equiv \Omega_k$ as

$$\mathcal{T}_k = \left(\begin{array}{c|c} t_{11} & t^\dagger \\ \hline t & \mathcal{T}_{k-1} \end{array}\right) \qquad \Omega_k = \left(\begin{array}{c|c} \omega_{11} & \omega^\dagger \\ \hline \omega & \Omega_{k-1} \end{array}\right). \tag{6}$$

We will derive a recurrence relation by first integrating over t_{11} and then over the vector t. To this end we use the identity

$$\det(1 + \mathrm{i}\mathcal{T}_k) = (1 + \mathrm{i}t_{11} + t^\dagger(1 + \mathrm{i}\mathcal{T}_{k-1})^{-1}t)\det(1 + \mathrm{i}\mathcal{T}_{k-1}). \tag{7}$$

The integral over t_{11} can be done by residues giving

$$\int \mathrm{d}\mathcal{T}_{k-1}\,\mathrm{e}^{\mathrm{i}\mathrm{Tr}\,\mathcal{T}_{k-1}\Omega_{k-1}}(\det(1 + \mathrm{i}\mathcal{T}_{k-1}))^{-n}\omega_{11}^{n-1}\,\mathrm{e}^{-\omega_{11}}\int \mathrm{d}t\,\mathrm{d}t^\dagger\,\mathrm{e}^{-\omega_{11}t^\dagger(1+\mathrm{i}\mathcal{T}_{k-1})^{-1}t}\,\mathrm{e}^{\mathrm{i}(t^\dagger\omega+\omega^\dagger t)}. \tag{8}$$

The last integral is Gaussian giving

$$\frac{1}{\omega_{11}^{k-1}}\det(1 + \mathrm{i}\mathcal{T}_{k-1})\,\mathrm{e}^{-\frac{1}{\omega_{11}}\omega^\dagger(1+\mathrm{i}\mathcal{T}_{k-1})\omega}. \tag{9}$$

Substituting this back into (8) leads to

$$\int \mathrm{d}\mathcal{T}_{k-1}(\det(1 + \mathrm{i}\mathcal{T}_{k-1}))^{-(n-1)}\,\mathrm{e}^{\mathrm{i}\mathrm{Tr}\,\mathcal{T}_{k-1}(\Omega_{k-1}-\frac{1}{\omega_{11}}\omega\omega^\dagger)}\omega_{11}^{n-k}\,\mathrm{e}^{-\omega_{11}-\frac{\omega^\dagger\omega}{\omega_{11}}}. \tag{10}$$

Hence we are led to the recurrence relation

$$P_{n,k}(\Omega_k) = C''_{n,k}\omega_{11}^{n-k}\,\mathrm{e}^{-\omega_{11}-\frac{\omega^\dagger\omega}{\omega_{11}}}P_{n-1,k-1}\left(\Omega_{k-1} - \frac{1}{\omega_{11}}\omega\omega^\dagger\right). \tag{11}$$

4. Probability distributions for elements of Ω

We will now use the recurrence relation (11) to determine $P_{n,k}(\Omega)$. Let us first consider the easy 'Wishart case' ($n \geqslant k$). Then repeated use of (11) reduces the problem to calculating $P_{n,1}(\Omega)$ which is just

$$P_{n,1}(\omega) = \omega^{n-1}\,\mathrm{e}^{-\omega}. \tag{12}$$

Since in the Wishart case the eigenvalues are generically distinct we may diagonalize the matrix Ω, and then the recurrence relation can be solved immediately [10] to get

$$P_{n,k}^{\mathrm{Wishart}}(\Omega) = C'''_{n,k}(\det\Omega)^{n-k}\,\mathrm{e}^{-\mathrm{Tr}\,\Omega}. \tag{13}$$

Let us now turn to the more interesting anti-Wishart case ($n < k$). Then repeated use of the recurrence relation reduces to the initial condition

$$P_{0,k}(\Omega_k) = \delta(\Omega_k). \tag{14}$$

$\delta(\Omega_k)$ is defined on the space of Hermitian matrices through the integral

$$\delta(\Omega_k) = \int \mathrm{d}\mathcal{A}\,\mathrm{e}^{\mathrm{i}\mathrm{Tr}\,\Omega_k\mathcal{A}} \tag{15}$$

where the integration domain covers the space of *Hermitian* $k \times k$ matrices.

In the first nontrivial case we then have

$$P_{1,k}(\Omega_k) = \omega_{11}^{1-k}\,\mathrm{e}^{-\omega_{11}-\frac{\omega^\dagger\omega}{\omega_{11}}}\,\delta\left(\Omega_{k-1} - \frac{1}{\omega_{11}}\omega\omega^\dagger\right). \tag{16}$$

Due to the form of the Dirac delta function we may immediately recognize that the argument of the exponent is just the ordinary trace of Ω_k.

4.1. Solution of the recurrence relations

We will now obtain an explicit expression for general $n < k$. The recurrence relation (11) expresses the probability distribution of a $k \times k$ matrix Ω_k by a probability distribution of a $(k-1)\times(k-1)$ matrix. The first step towards finding the general solution is to obtain an explicit expression for the elements of the relevant $(k-i)\times(k-i)$ matrix at the ith step of the recursion.

We therefore have to solve

$$\Omega^{(k)} = \Omega \tag{17}$$

$$\Omega^{(i-1)} = \Omega^{(i)}_{i-1} - \frac{\omega^{(i)}\omega^{(i)\dagger}}{\omega^{(i)}_{11}} \tag{18}$$

where the superscript (i) denotes the $(k-i)$th step of the recursion while the subscripts are defined through the decomposition analogous to (6):

$$\Omega^{(i)} = \left(\begin{array}{c|c} \omega^{(i)}_{11} & \omega^{(i)\dagger} \\ \hline \omega^{(i)} & \Omega^{(i)}_{i-1} \end{array}\right). \tag{19}$$

The 'reduced' matrix $\Omega^{(i)}$ is of size $i \times i$.

Once the explicit expressions for $\Omega^{(i)}$ are known, the probability distribution for the anti-Wishart case can be written as

$$P_{n,k}(\Omega_k) \propto \left[\prod_{i=0}^{n-1}\omega_{11}^{(k-i)}\right]^{n-k} \exp\left(-\sum_{i=0}^{n-1}\left\{\omega_{11}^{(k-i)} + \frac{\omega^{(k-i)\dagger}\omega^{(k-i)}}{\omega_{11}^{(k-i)}}\right\}\right)\delta(\Omega^{(k-n)}). \tag{20}$$

The matrix-valued delta in (20) of the Hermitian $(k-n)\times(k-n)$ matrix $\Omega^{(k-n)}$ is equivalent to the set of $(k-n)^2$ deltas for the independent matrix elements (keeping in mind that the diagonal elements are real and the off-diagonal ones appear in complex conjugate pairs).

We will first show that the argument of the exponent is just the trace of Ω. Let us take the trace of both sides of the recurrence relation (18):

$$\mathrm{Tr}\,\Omega^{(i-1)} = \mathrm{Tr}\,\Omega^{(i)} - \omega^{(i)}_{11} - \frac{\omega^{(i)\dagger}\omega^{(i)}}{\omega^{(i)}_{11}}. \tag{21}$$

Then the sum in the exponent may be rewritten as

$$-\sum_{i=0}^{n-1}(\mathrm{Tr}\,\Omega^{(k-i)} - \mathrm{Tr}\,\Omega^{(k-i-1)}). \tag{22}$$

In this alternating sum all terms cancel except the first and the last which give $-\mathrm{Tr}\,\Omega^{(k)} + \mathrm{Tr}\,\Omega^{(k-n)}$. The last matrix vanishes due to the delta function in (20), while the first one is just the original matrix Ω (see (17)).

In order to deal with the remaining terms in (20) we have to explicitly solve the recurrence relations (17)–(18).

Remarkably enough, one can give a compact formula for the reduced matrices in terms of a ratio of determinants. Let us denote by $\Omega_{[i]}$ the upper left hand $i \times i$ sub-matrix of the *original* matrix Ω:

$$\Omega = \left(\begin{array}{c|c} \Omega_{[i]} & * \\ \hline * & * \end{array} \right). \tag{23}$$

Furthermore, for each $l, m > i$ we will consider the $(i+1) \times (i+1)$ matrix $\Omega_{[i],lm}$ obtained by adjoining the lth row and mth column of Ω to $\Omega_{[i]}$:

$$\Omega_{[i],lm} = \left(\begin{array}{ccc|c} & & & \omega_{1m} \\ & \Omega_{[i]} & & \cdots \\ & & & \omega_{im} \\ \hline \omega_{l1} & \cdots & \omega_{li} & \omega_{lm} \end{array} \right). \tag{24}$$

In terms of these data, the solution to (17) can be expressed through the simple formula

$$\Omega_{lm}^{(k-i)} = \frac{\det \Omega_{[i],l+im+i}}{\det \Omega_{[i]}}. \tag{25}$$

We give some details of the proof in appendix B. From here we can easily read off the elements $\omega_{11}^{(i)}$ entering formula (20):

$$\omega_{11}^{(k-i)} = \frac{\det \Omega_{[i+1]}}{\det \Omega_{[i]}} \tag{26}$$

where $\det \Omega_{[0]}$ is understood as 1. Putting together the above results, we obtain our final result for the probability distribution for the anti-Wishart case:

$$P_{n,k}^{\text{anti-Wishart}}(\Omega) = C_{n,k}^{''''} (\det \Omega_{[n]})^{n-k} \, \mathrm{e}^{-\mathrm{Tr}\,\Omega} \prod_{l=n+1}^{k} \prod_{m=n+1}^{k} \delta\left(\frac{\det \Omega_{[n],lm}}{\det \Omega_{[n]}} \right). \tag{27}$$

As explained above, the number of independent deltas is $(k-n)^2$.

The above formula has two important features. The Dirac delta function shows that part of the matrix elements of Ω are non-random and are expressed deterministically in terms of the first n rows of Ω. In fact this does not depend in any way on the type of randomness assumed. The reconstruction formula works even for a fixed $\Omega = \mathcal{A}^\dagger \mathcal{A}$. Moreover the quotient of the determinants is a *linear* function of ω_{lm}, thus giving a simple expression for ω_{lm} in terms of the elements coming from the first n rows of Ω. In addition we note that given a fixed matrix $\Omega = \mathcal{A}^\dagger \mathcal{A}$ empirically, with absolutely no information on $\mathcal{A}$, we may determine the size of the $\mathcal{A}$ matrix by computing the successive reduced matrices $\Omega^{(i)}$, and checking the value of i when these matrices vanish.

The pre-factor of the Dirac delta function, which gives the probability distribution for the first n rows of Ω is linked with the type of randomness of $\mathcal{A}$.

5. Joint eigenvalue distributions

In this section, for completeness, we give the results for the joint eigenvalue distributions for Wishart and anti-Wishart random matrices, which, in contrast to the probability distributions of the matrix elements themselves $P_{n,k}(\Omega_k)$ derived in the preceding section, are rather well known[3] [11]. In the Wishart case ($n \geqslant k$) all the eigenvalues are generically distinct and the result follows from (13) in the standard way by including a Vandermonde determinant [12]:

$$P_{n,k}^{\text{Wishart}}(\lambda_1, \ldots, \lambda_k) = \prod_i \lambda_i^{n-k} \, \mathrm{e}^{-\lambda_i} \prod_{i<j} (\lambda_i - \lambda_j)^2. \tag{28}$$

[3] For similar results for the real case, see Hsu [11] or simultaneous independent papers by each of Fisher, Girshick, Mood and Roy in [2].

In the anti-Wishart case there are always $k - n$ exact zero-modes. The remaining n nonzero eigenvalues of $\mathcal{A}^\dagger\mathcal{A}$ are distributed with the *same* probability distribution as the eigenvalues of $\mathcal{A}\mathcal{A}^\dagger$. This can be seen by diagonalizing the rectangular matrix $\mathcal{A}$ through $\mathcal{A} = U\mathcal{S}V$ with $U \in U(k)$, $V \in U(n)$ and

$$\mathcal{S} = \begin{pmatrix} \Lambda \\ 0 \end{pmatrix} \tag{29}$$

is a $k \times n$ matrix with Λ diagonal. Then $\mathcal{A}\mathcal{A}^\dagger = U\mathcal{S}\mathcal{S}^\dagger U^\dagger$ and $\mathcal{A}^\dagger\mathcal{A} = V^\dagger\mathcal{S}^\dagger\mathcal{S}V$. Hence the nonzero eigenvalues of $\mathcal{A}\mathcal{A}^\dagger$ and $\mathcal{A}^\dagger\mathcal{A}$ coincide.

We may then write immediately the joint eigenvalue distribution for the anti-Wishart case $(n < k)$:

$$P_{n,k}^{\text{anti-Wishart}}(\lambda_1, \ldots, \lambda_n) = \prod_i \lambda_i^{k-n}\, \mathrm{e}^{-\lambda_i} \prod_{i<j} (\lambda_i - \lambda_j)^2. \tag{30}$$

We do not discuss here the asymptotic spectral properties of (28) and (30). For recent results and original references we refer to the recent mathematical literature on this subject [13].

6. Conclusions

Using the recently advocated properties of the Ingham–Siegel [10] integrals, we have provided a compact expression for the distribution of the matrix elements of the Wishart ensemble, in the case where the numbers of rows and columns are finite and arbitrary. We gave a unified derivation which encompasses both the classical Wishart case (where the number of rows is greater or equal to the number of columns), and the opposite case, where part of the matrix is necessarily non-random (anti-Wishart case).

The expression obtained is a starting point for algorithms leading to fast reconstruction of the redundant information from the first n rows of the Wishart matrix. The formulae for the reduced matrices allow us also to test an empirical Wishart-like matrix for redundant information. The more general problem of reconstructing the Wishart matrix from less regularly distributed data points (sparse matrices) remains a challenging problem, which will be analysed elsewhere.

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Note added. The number of constraints for the real anti-Wishart distribution is *the same* as in [15]. The apparent discrepancy in the first version of this manuscript was only due to a sloppy rewriting of a matrix-valued Dirac delta function in terms of component scalar Dirac delta functions.

Appendix A. Real Wishart and anti-Wishart matrices

In this section, we will briefly summarize the relevant formulae for $\Omega = \mathcal{A}^\dagger\mathcal{A}$ where now $\mathcal{A}$ is taken from a Gaussian *real* ensemble. The determination of $P(\Omega)$ follows the complex case, using now the integral representation

$$P_{n,k}(\Omega) = \int \mathrm{d}\mathcal{T}\, \det(1 + \mathrm{i}\mathcal{T})^{-n/2}\, \mathrm{e}^{\mathrm{i}\,\mathrm{Tr}\,\mathcal{T}\Omega} \tag{31}$$

where we now integrate over *real symmetric* matrices $\mathcal{T}$. Instead of the integration by residues leading to (8), we use the formula [14]

$$\int_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}tw} \frac{1}{(a+\mathrm{i}t)^{\nu}} = \frac{1}{\Gamma(\nu)} 2\pi w^{\nu-1}\, \mathrm{e}^{-wa}. \tag{32}$$

Integrating over t_{11} gives, modulo overall constants,

$$\int \mathrm{d}\mathcal{T}_{k-1}\, \mathrm{e}^{\mathrm{i}\operatorname{Tr}\mathcal{T}_{k-1}\Omega_{k-1}} (\det(1+\mathrm{i}\mathcal{T}_{k-1}))^{-n/2} \omega_{11}^{n/2-1}\, \mathrm{e}^{-\omega_{11}} \int \mathrm{d}t\, \mathrm{d}t^{\dagger}\, \mathrm{e}^{-\omega_{11} t^{\dagger}(1+\mathrm{i}\mathcal{T}_{k-1})^{-1} t}\, \mathrm{e}^{\mathrm{i}(t^{\dagger}\omega+\omega^{\dagger}t)}. \tag{33}$$

The last integral is real Gaussian giving

$$\frac{1}{\omega_{11}^{(k-1)/2}} [\det(1+\mathrm{i}\mathcal{T}_{k-1})]^{\frac{1}{2}}\, \mathrm{e}^{-\frac{1}{\omega_{11}}\omega^{\dagger}(1+\mathrm{i}\mathcal{T}_{k-1})\omega}. \tag{34}$$

Hence we are led to the recurrence relation

$$P_{n,k}(\Omega_k) \propto \omega_{11}^{\frac{n-k-1}{2}}\, \mathrm{e}^{-\omega_{11}-\frac{\omega^{\dagger}\omega}{\omega_{11}}} P_{n-1,k-1}\left(\Omega_{k-1} - \frac{1}{\omega_{11}}\omega\omega^{\dagger}\right). \tag{35}$$

In the classical Wishart case (real variables) using relation (35) repeatedly we may reduce the problem to calculating $P_{n,1}(\Omega) = \omega^{n/2-1}\,\mathrm{e}^{-\omega}$. Again the recurrence relation may be solved after diagonalizing Ω, and we recover the result

$$P_{n,k}^{\text{Wishart}}(\Omega) = C''''_{n,k} (\det\Omega)^{\frac{n-k-1}{2}}\, \mathrm{e}^{-\operatorname{Tr}\Omega}. \tag{36}$$

Similar iteration as in the complex anti-Wishart case (19) leads to the matrix-valued delta for the reduced matrix $\Omega^{(k-n)}$. It is defined by a formula analogous to (15) but with the domain of integration being restricted to the space of *real symmetric* $(k-n)\times(k-n)$ matrices. Its $(k-n)(k-n+1)/2$ independent elements can be again rewritten as ratios of determinants, thus arriving at the probability distribution

$$P_{n,k}^{\text{anti-Wishart}}(\Omega) = C''''_{n,k} (\det\Omega_{[n]})^{\frac{n-k-1}{2}}\, \mathrm{e}^{-\operatorname{Tr}\Omega} \prod_{l=n+1}^{k} \prod_{\substack{m=n+1\\ l\leqslant m}}^{k} \delta\left(\frac{\det\Omega_{[n],lm}}{\det\Omega_{[n]}}\right). \tag{37}$$

The number of independent scalar deltas is equal to $(k-n)(k-n+1)/2$ and agrees with that in [15].

For completeness, we recall that joint eigenvalue distributions follow once we include the appropriate real Vandermonde determinants:

$$P_{n,k}^{\text{Wishart}}(\lambda_1,\ldots,\lambda_k) = \prod_i \lambda_i^{\frac{n-k-1}{2}}\, \mathrm{e}^{-\lambda_i} \prod_{i<j} (\lambda_i-\lambda_j)^{\beta} \tag{38}$$

$$P_{n,k}^{\text{anti -Wishart}}(\lambda_1,\ldots,\lambda_n) = \prod_i \lambda_i^{\frac{k-n-1}{2}}\, \mathrm{e}^{-\lambda_i} \prod_{i<j} (\lambda_i-\lambda_j)^{\beta} \tag{39}$$

where $\beta = 1$ for the real case. For recent results on spectral analysis of the above formula we refer to [13].

Appendix B. Some details of the proof of (25)

The proof can be done by induction on i. For $i = 0$ the statement is obviously true. Since in this case we have

$$\Omega_{lm}^{(k)} = \frac{\det \Omega_{[0],lm}}{\det \Omega_{[0]}} = \Omega_{lm}. \tag{40}$$

Let us assume that formula (25) holds for $i - 1$. We will verify the formula for i. To this end let us rewrite the recurrence relation (18)

$$\Omega_{lm}^{(k-i)} = \Omega_{k-i;lm}^{(k-i+1)} - \frac{\omega^{(k-i+1)}\omega^{(k-i+1)\dagger}}{\omega_{11}^{(k-i+1)}} \tag{41}$$

as

$$\Omega_{lm}^{(k-i)} = \Omega_{l+1\,m+1}^{(k-i+1)} - \frac{\Omega_{l+11}^{(k-i+1)}\Omega_{1m+1}^{(k-i+1)}}{\Omega_{11}^{(k-i+1)}} \tag{42}$$

where we made use of the decomposition (6). We will now insert formula (25) to obtain

$$\frac{\det \Omega_{[i]l+im+i}}{\det \Omega_{[i]}} = \frac{\det \Omega_{[i-1]l+im+i}}{\det \Omega_{[i-1]}} - \frac{\det \Omega_{[i-1]l+ii} \det \Omega_{[i-1]im+i}}{\det \Omega_{[i-1]} \det \Omega_{[i-1]ii}}. \tag{43}$$

When we multiply this by $\det \Omega_{[i]} \det \Omega_{[i-1]}$ we see that the induction step is equivalent to proving the following determinental identity[4]:

$$\det \Omega_{[i-1]} \det \Omega_{[i]l+im+i} = \det \Omega_{[i-1]l+im+i} \det \Omega_{[i-1]ii} - \det \Omega_{[i-1]l+ii} \det \Omega_{[i-1]im+i} \tag{44}$$

where we used the notation defined in (23) and (24). Now we use repeatedly, for each of the terms of (44), the decomposition

$$\det \left(\begin{array}{c|c} \Omega_{[i-1]} & \psi \\ \hline \psi^\dagger & \sigma \end{array} \right) = \det \Omega_{[i-1]} \left(\sigma - \psi^\dagger \Omega_{[i-1]}^{-1} \psi \right). \tag{45}$$

The only complication lies in applying this formula to $\det \Omega_{[i+1]}$ where terms involving $\Omega_{[i]}^{-1}$ do appear. To proceed we have to use a similar decomposition for the inverse matrix:

$$\Omega_{[i]}^{-1} \equiv \left(\begin{array}{c|c} \Omega_{[i-1]} & \psi \\ \hline \psi^\dagger & \sigma \end{array} \right)^{-1} = \left(\begin{array}{c|c} A & b \\ \hline b^\dagger & c \end{array} \right) \tag{46}$$

with

$$A = \Omega_{[i-1]}^{-1} - \Omega_{[i-1]}^{-1} \psi b^\dagger \tag{47}$$

$$b = -c\Omega_{[i-1]}^{-1} \psi \tag{48}$$

$$b^\dagger = -c\psi^\dagger \Omega_{[i-1]}^{-1} \tag{49}$$

$$c = \frac{1}{\sigma - \psi^\dagger \Omega_{[i-1]}^{-1} \psi}. \tag{50}$$

After a straightforward but tedious calculation we arrive at (44).

[4] Note that $\det \Omega_{[i-1]i,i} = \det \Omega_{[i]}$.

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